The log world

We shall give three equivalent definitions of of divisors with normal crossings respectively simple normal crossings

X = variety , $D = \sum_{i=1}^{N} D_i$ Weil divisor , $\dim X = n$

1. D has normal crossings (nc) respectively simple normal crossings (snc)

iff for all $x \in X$ (closed point) there exists a regular system of parameters

$z_1, \ldots, z_n \in \widehat{\mathcal{O}}_{X,x}$ (for nc) resp. $z_1, \ldots, z_n \in \mathcal{O}_{X,x}$ (for snc)

s.t. $f := z_1 \cdot \ldots \cdot z_k$ is an equation for D in x

(Here, k depends on $x \in X$)

2. D has snc if X is smooth along D (i.e. along $\operatorname{supp}(D)$) and s.t. $\bigcap_{j \in I(x)} D_j$ is smooth of codimension $|I(x)|$

for all $x \in X$. Here, $I(x) = \{ j \mid x \in D_j \}$

D has only nc if (X, D) satisfies the condition locally in the étale topology.

3. In the analytic situation: X has nc resp. snc if X is smooth along D and for all $x \in X$ there exists an analytic coordinate system $z_1, \ldots, z_n : U \to \mathbb{C}^n$ with $x \in U \subset X$ open s.t. $D \cap U = (z_1 \cdot \ldots \cdot z_k)$ resp. $D_{j_i} = (z_i)$ for $I(x) = \{ j_1, \ldots, j_k \}$

Remarks / Examples

- For simplicity one usually works with snc, although nc would suffice
- Roughly, snc = nc + each D_i smooth
-  would be nc, but not snc
- Since X is smooth along D, the divisor D is in fact Cartier.

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In order to motivate the definition of the logarithmic canonical bundle, we shall give the various equivalent definitions of differentials with log poles. (This will not strictly be needed later.)

Easy case: $D \subset X$ a smooth irreducible divisor in a smooth variety

$\Omega_X(\log D)$ is the locally free sheaf of differential forms with log poles along D, which can be defined as follows

local definition: $z_1, \dots z_n$ local parameter around $x \in D \subset X$ s.t. $D = (z_1)$. Then $\Omega_X(\log D) = \langle \frac{dz_1}{z_1}, dz_2, \dots dz_n \rangle$, i.e. locally $\Omega_X(\log D)$ is freely generated as $\mathcal{O}_X$-module by $\frac{dz_1}{z_1}, dz_2, \dots, dz_n$. By definition $\Omega_X(\log D)|_{X \setminus D} = \Omega_X$.

via restriction: Consider the restriction of forms $\alpha \mapsto \alpha|_D$ as a morphism of sheaves on X.

$$\text{restr}: \Omega_X \longrightarrow \Omega_D$$

Then $\Omega_X(\log D) = \operatorname{Ker}(\mathrm{restr}) \otimes \mathcal{O}(D)$. (*)

(Locally Ω_X generated by $dz_1, \dots dz_n$ and

Ω_D generated by $dz_{2|D} \dots dz_{n|D}$

Hence, $\operatorname{Ker}(\mathrm{restr})$ is generated by $dz_1, z_1 \cdot dz_2, \dots z_1 dz_n$.

Then use that $\mathcal{O}(D)$ is generated by $1/z_1$.)

<u>via the residue</u>: There exists a short exact sequence

$$0 \to \Omega_X \to \Omega_X(\log D) \xrightarrow{\mathrm{res}_D} \mathcal{O}_D \to 0 \qquad (**)$$

$$\alpha = \sum \alpha_i dz_i \longmapsto \mathrm{res}_D(\alpha) = z_1 \cdot \alpha_1$$

Higher order: $\Omega_X^i(\log D) := \Lambda^i(\Omega_X(\log D))$

Remark: If $j : X \setminus D \hookrightarrow X$ and $\Omega_X^\bullet(*D) := j_* \Omega_{X\setminus D}^\bullet$

("algebraic de Rham complex"), then

$\Omega_X^\bullet(\log D) \subset \Omega_X^\bullet(*D)$ is a subcomplex

The descriptions of $\Omega_X(\log D)$ given above lead to short exact sequences

$$0 \to \Omega_X^i(\log D) \otimes \mathcal{O}(-D) \to \Omega_X^i \to \Omega_D^i \to 0 \qquad \text{(restriction)}$$

and

$$0 \to \Omega_X^i \to \Omega_X^i(\log D) \to \Omega_D^{i-1} \to 0 \qquad \text{(residue)}$$

Locally $\Omega_X^i(\log D)$ freely generated by

$$\frac{dz_1}{z_1} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_{i-1}}, \quad dz_{j_1} \wedge \dots \wedge dz_{j_i}$$

$$j_k \in \{2, \dots n\}$$

The *logarithmic canonical bundle* is then naturally defined as

$$\Omega_X^n(\log D) = \det(\Omega_X(\log D))$$

which is simply $\underline{\Omega_X^n(\log D) = K_X + D}$

General case Let $D = \sum^N D_i$ be a SNC divisor.

If D is locally around $x \in D$ given by $z_1 \cdots z_k = 0$ where $z_1 \ldots z_n \in \mathcal{O}_{X,x}$ regular system of parameters, then

$\Omega_X(\log D)$ freely generated by

$$\frac{dz_1}{z_1}, \ldots \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$$

In analogy to the above descriptions one gets short exact sequences

$$0 \to \Omega_X(\log D)(-D_1) \to \Omega_X(\log(\sum_{i=2}^N D_i)) \to \Omega_{D_1}(\log(\sum_{i=2}^N D_i)|_{D_1}) \to 0$$

and then go on by recursion. Note that $\sum_{i=2}^N D_i|_{D_1} \subset D_1$ is SNC. Taking higher exterior powers.

$$0 \to \Omega_X^i(\log D)(-D_1) \to \Omega_X^i(\log(\sum_{i=2}^N D_i)) \to \Omega_{D_1}^i(\log(\sum_{i=2}^N D_i)|_{D_1}) \to 0$$

and eventually $\det(\Omega_X(\log D)) = K_X + D$

the logarithmic canonical bundle

(A priori this makes sense for X smooth, but can be taken as the definition e.g. for X Gorenstein or even $\mathbb{Q}$-Gorenstein.)

The residue defines that exact sequences

$$0 \to \Omega_X \to \Omega_X(\log D) \to \bigoplus \mathcal{O}_{D_i} \to 0$$

and

$$0 \to \Omega_X(\log(\sum_{i=2}^{N} D_i)) \to \Omega_X(\log D) \to \mathcal{O}_{D_1} \to 0 .$$

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A *log pair* (X, D) consists of a

- normal variety X
- a $\mathbb{Q}$-Weil divisor $D = \sum d_i D_i$ with $0 \le d_i \le 1$.

Rem. Sometimes one requires $0 \le d_i < 1$ or only $0 \le d_i$. We shall comment on this later.

The *log canonical divisor* of (X, D) is

$$K_X + D \in Z(X)_{\mathbb{Q}}.$$

Suppose $f : Y \to X$ is proper, birational

$$\rightsquigarrow \quad D_Y := \tilde{D} + \sum E_i ,$$

where $\tilde{D} := \sum d_i \tilde{D}_i$ is the strict transform of D and $E_1, \dots, E_k$ are the exceptional divisors of f

One considers $f : Y \to X$ as a morphism $(Y, D_Y) \to (X, D)$ of log pairs. (Sometimes one writes $\tilde{D} = f_*^{-1} D$)

If $K_X + D$ is $\mathbb{Q}$-Cartier (usually X $\mathbb{Q}$-Gorenstein and D $\mathbb{Q}$-Cartier), then one has the *log ramification formula*:

$$\underbrace{K_Y + D_Y}_{\substack{\text{log canonical}\\ \text{of } (Y, D_Y)}} = f^*\underbrace{(K_X + D)}_{\substack{\text{log canonical}\\ \text{of } (X, D)}} + \sum a_i E_i$$

Def. A log pair (X, D) has <u>log canonical singularities</u> iff i) $K_X + D$ is $\mathbb{Q}$-Cartier and

ii) $\exists f : Y \to X$ birational, proper, Y smooth

s.t. D_Y is nc and in the log ramification formula

$$K_Y + D_Y = f^*(K_X + D) + \sum_{i=1}^{k} a_i E_i \quad \text{one has}$$

$a_i \geq 0$.

<u>Remark</u>: • There is an equivalent way of expressing this:

Write the log ramification formula rather as

$$K_Y = f^*(K_X + D) + \sum b_i F_i ,$$

with $F_1 = E_1, \dots, F_k = E_k$, $F_{k+1} = \widehat{D}_1, \dots, F_{k+N} = \widehat{D}_N$

Then $b_i = a_i - 1$ for $i = 1, \dots, k$

$b_{k+i} = -d_i$ for $i = 1, \dots, N$.

Condition ii) could be replaced by the equivalent condition

ii)' $\exists f : Y \to X$ birational, proper, Y smooth

s.t. $\sum F_i$ is nc and $b_i \geq -1$ for any

(<u>exceptional</u>) F_i.

• As in the absolute case, if ii)' holds for one resolution then it holds for any. The proof is principally the same, but one has to be more precise throughout. Here are the details: As in the absolute case, it suffices to consider

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[rr, bend right, "f'"'] & Y \arrow[r, "f"] & X \end{tikzcd}$$

Then one has $K_Y = f^*(K_X + D) + \sum b_i F_i$ (*)

and $K_{Y'} = f'^*(K_X + D) + \sum b_j' F_j'$

and one has to show

a) $\forall i$: $b_i \geq -1$ for F_i exceptional,

$\Longleftrightarrow$

b) $\forall j$: $b_j' \geq -1$ for F_j' exceptional

Pulling back (*) yields

$$K_{Y'} = g^* K_Y + \sum c_e G_e \qquad G_e \text{ are the } g\text{-except. div.}$$

$$= f'^*(K_X + D) + g^*\left(\sum b_i F_i\right) + \sum c_e G_e$$

Write $g^* F_i = \widetilde{F_i} + \sum c_{ie} G_e$. Hence

$$K_{Y'} = f'^*(K_X + D) + \sum_e \left(\sum_i b_i c_{ie} + c_e \right) G_e + \sum b_i \widetilde{F_i}$$

Since $\widetilde{F_i}$ except. for f' iff F_i is f except., it suffices to show that $\sum_i b_i c_{ie} + c_e \geq -1$ if a) holds.

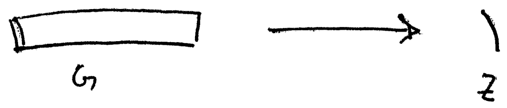
Earlier we only had to use $c_e > 0$ for $g: Y' \to Y$ with Y, Y' smooth. This is not quite enough here.

For simplicity we shall first assume that $\sum F_i$ is snc and $g: Y' \to Y$ is a single blow-up of Y along an irred. (smooth) $Z \subset Y$. Then $K_{Y'} = g^* K_Y + c \cdot G$ with $c = \operatorname{codim}(Z) - 1$. (Use that the cokernel of $J_{Y'} \to g^* J_Y$ is a locally free sheaf of rank $= c$ on G.)

Moreover, if $g^* F_i = \widehat{F_i} + c_{i1} G$, then

$c_{i1} = 0$ if $Z \not\subset F_i$,

$c_{i1} = 1$ if $Z \subset F_i$ (F_i smooth)



Hence $\sum_i b_i c_{i1} + c$

$$= \sum_{\substack{i \\ Z \subset F_i}} b_i + \operatorname{codim}(Z) - 1$$

As $\sum F_i$ is snc, $\# \{ i \mid Z \subset F_i \} \leq \operatorname{codim}(Z)$.

a) $\Rightarrow$ $b_i \geq -1$ for F_i exceptional, but as $D = \sum d_i D_i$ with $\underline{\underline{d_i \leq 1}}$, one has in fact $b_i \geq -1$ for all F_i.

Thus $\sum_i b_i c_{i1} + c \geq -\operatorname{codim}(Z) + \operatorname{codim}(Z) - 1 = -1$.

The assumption $\sum F_i$ <u>snc</u> is superfluous, <u>nc</u> would suffice for the above argument. (Consider a local analytic nbhd of some general point of Z.)

For the general case one may use the

<u>Lemma</u>: Let Z, Y be smooth D_Z, D_Y divisors on Z resp. Y. If D_Y has nc and $f^{-1}(D_Y) \subset D_Z$, then

$$K_Z + D_Z = f^*(K_Y + D_Y) + E \text{ for some}$$

effective divisor E.

In our case set $Z = Y'$, $f = g$, $D_Z = \left(g^* \sum_{b_i<0} F_i\right)_{red}$

$$D_Y = \sum_{b_i<0} F_i$$

$$\Rightarrow K_{Y'} = g^*\Big(K_Y + \sum_{b_i<0} F_1\Big) - \Big(g^* \sum_{b_i<0} F_i\Big)_{red} + E$$

$$= g^* f^*(K_X + D) + g^* \sum_{b_i \geq 0} b_i F_i + g^* \sum_{b_i<0} (1+b_i) F_i - \Big(g^* \sum_{b_i<0} F_i\Big)_{red} + E$$

This is enough to conclude ($\to$ KMM)