Moduli spaces of Higgs bundles and the Hitchin system

Winter term 2017/18, Tuesday 2-4pm, 0.011

The following is a list of topics that we should try to cover during the semester. Not all of the talks will fit in one session. Others will be survey talks without technical details. The first few should come in this order, the later ones are more independent. The subject is vast, so we will unfortunately not have the time to build up the theory methodically. The hope is that the seminar is accessible to people without any prior knowledge in moduli space of bundles on curves, although they will have to take a few things for granted, but also contains parts that will be exciting for hard core experts.

To get a quick idea what this is all about check [22] (or the shorter and rougher original notes [19]).

Please contact me (huybrech@math.uni-bonn.de) for further information and if you want to give a talk. Suggestions on the program are very welcome.

The dates are provisional. It could very well be that some of the topics cannot be covered in just one talk.

10.10.2017: Moduli stacks and spaces of bundles on curves. (Speaker: Daniel Huybrechts)
Quickly recall the notion of (semi)stability for bundles on curves and introduce moduli spaces of (semi)stable bundles (with fixed determinant). State existence results as well as criteria for projectivity and smoothness. Give the dimension formula and describe the Picard group [13]. We will also need the case of moduli space of PGL_n-bundles, which is just a quotient of the moduli space of bundles. See [22, Sec. 2.2], [36, Sec. 2] or [42] for surveys of the main features and e.g. [15, Sec. 4] or [26, Lec. 3] for comments in the comparison between moduli stacks and moduli spaces.

17.10.2017: Moduli of Higgs bundles, spectral curves, and the Hitchin map. (Speaker: Bingyu Xia)
Define Higgs bundle and explain how to view them as direct images of torsion free sheaves on the spectral cover. Introduce the notion of stability of Higgs bundles. State the existence of the moduli space and construct the Hitchin fibration and explain that it is Lagrangian and proper. Describe the generic fibre as the Jacobian of the corresponding spectral curve. See [1, 37] and [38, Sec. 1-2]. See [22, Sec. 2.2-2.4, 3.1-3.2], and [24, Sec. 1] for surveys. Of course, it all started with [27].

7.11.2017: Cohomology of the Higgs moduli space. (Speaker: Luca Tasin)
Follow the survey [24, Sec. 1] (unfortunately, some references are missing) to show that the cohomology of the Higgs moduli space (in the coprime case) is pure and concentrated below the middle cohomology. Then switch to [21] where it is shown that $H^*_c \to H^*$ is trivial for the moduli space of rank two Higgs bundles of odd degree. In particular, all intersection numbers are trivial. It was conjectured (and proved in

the rank two case) in [23] that this holds true always for PGL_n-Higgs bundles of odd degree. The conjecture was recently verified in [25].

14.11.2017: Moduli spaces of stable sheaves on (K3) surfaces. (Speaker: Emma Brakkee)
This talk should survey the theory of moduli spaces of stable sheaves on surfaces. In later talks we will use this to view the moduli space of Higgs bundles as an open subset of a moduli space of torsion sheaves on the projectivized cotangent bundle and as degeneration of the moduli space of sheaves on K3 surfaces. In particular, explain that on a K3 surface the moduli space is symplectic and that for torsion sheaves with non-trivial determinant the moduli space admits a Lagrangian fibration over a complete linear system. See [28, Ch. 10] for references. Mention that in the case that semistability implies stabililty the cohomology of the moduli spaces of sheaves on K3 surfaces is completely known (and isomorphic to the cohomology of the Hilbert scheme).

21.11.2017: Higgs moduli space vs moduli space of stable sheaves on the cotangent bundle. (Speaker: Isabell Grosse-Brauckmann)
Via the spectral curve picture, the moduli space of Higgs bundles can be viewed as a moduli space of stable torsion sheaves (concentrated on the spectral curves) on the cotangent bundle of the curve. This has two nice features. First, the moduli space can then be naturally compactified to the moduli space of stable sheaves on a compactification of the cotangent bundle (which still comes with a Hitchin map, but cannot be symplectic!). Second, it can often be seen as a degeneration of the moduli space of stable torsion sheaves on a K3 surface (this only works in the case of fixed degree but not for fixed determinant). Explain this point of view following [10]. It will be useful to add information on the nilpotent cone following [29] or [14].

28.11.2017: Generators of the cohomology. (Speaker: Torsten Beckmann/Tim Büllles)
Follow [30, Sec. 2, 4] and prove that the components of the universal family of Higgs bundles generate the cohomology algebra of the moduli space (in the coprime case). Markman's approach is via moduli spaces of stable sheaves on surfaces. The result is originally due to Hausel and Thaddeus [18] (the published version only treats the rank two case). There won't be enough time to go through all the computations in [30], but it should be possible to give an outline. The paper relies on a result of Hausel [17] that shows that all cohomology comes from the natural compactification. It could be a good idea to start with Beauville's proof [2] of a classical theorem of Atiyah and Bott proving a similar statement for the cohomology of the moduli space of stable bundles (coprime case), cf. [42]. (The case of fixed determinant goes back to Newstead.)

Wednesday(!!!) 6.12.2017 (!!!): Conjecture of Hausel– Rodriguez-Villegas. (Guest speaker: Jochen Heinloth)
This is devoted to the proof, following [25], of the conjecture saying that the intersection product on the moduli space of PGL Higgs bundles is trivial.

12.12.2017: Torelli theorem. (Speaker: Isabell Grosse-Brauckmann)
The standard Torelli theorem for curves, that the polarized Jacobian determines the

curve, has been generalized to moduli spaces of higher rank bundles. For the moduli space of stable bundle in [31, 32, 33] and, more recently, for the moduli space of Higgs bundles (coprime case) in [3], see also [4, 5]. The extension to Higgs bundles uses the Hitchin system and should be the main focus of the talk. It would however be good (and time should permit this) to also explain the original argument of Mumford and Newstead. See also [36, Sec. 4.4]

19.12.2017: Character variety and the fundamental theorem of non-abelian Hodge theory. (Speaker: Andrey Soldatenkov)
Explain the hyperkähler structure of the moduli spaces via Simpson's λ-connections. The main result is that the moduli space of Higgs bundles is diffeomorphic to the (affine!) variety of representations of the fundamental group. This builds up on the classical results due to Weil and Narasimhan–Seshadri [34] which describes a bijection between stable bundles and irreducible unitary representations. See [40], [22, Sec. 4], and [42, 41] for surveys and references.

9.1.2017: Mirror Symmetry and Langlands duality. (Speaker: Hacen Zelaci)
In this talk we should learn about the result of Hausel and Thaddeus [20] that relates the Higgs bundle moduli space for SL and PGL (which are Langlands dual groups). The first step is to explain that the (generic) fibres of the two Hitchin system are naturally (torsors of) dual abelian varieties. The second part should explain [22, Conj. 3.23]. See [22, Sec. 3.4,3.5,3.7] for a survey. Check out [16]. For generalizations see [11] and a discussion from the Langlands view point see [12]. For the analogous picture in the context of character varieties check out [22, Sec. 4].

16.1.2017: The $P = W$ conjecture. (Speaker: DH ?)
See [7, 8, 9] and also [22, Sec. 5.1]. It would be good to get a feeling for this conjecture, but it may turn out to be just too much. But maybe someone feels up to it?

References

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